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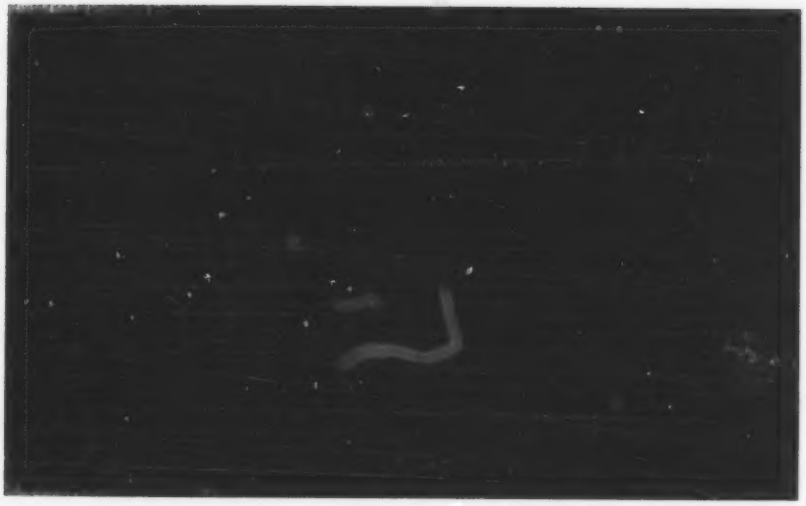

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Simplicial Decomposition Methods for the Transit Equilibrium Assignment Problem

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ABSTRACT

In this paper we consider a transit equilibrium assignment problem which is defined as assigning the passenger flow on a congested transit network (public transportation network) with asymmetric cost functions, given a fixed origin-destination matrix. The original problem in the hyperpath flow space is transformed into an equivalent problem in the space of total arc flows plus an auxiliary variable. A simplicial decomposition algorithm is developed and its convergence is given under usual assumptions. The algorithm requires much less memory and has high computational efficiency.

Keywords: Transit equilibrium assignment, Variational Inequalities, Simplicial decomposition.

RÉSUMÉ

Dans cet article nous considérons le problème d'affectation d'équilibre dans un réseau de transport en commun avec des fonctions de coût asymétriques et une demande fixe. Le problème original dans l'espace des hyperchemins est transformé en un problème équivalent dans l'espace des flux des arcs et une variable auxiliaire. Un algorithme de restriction simpliciale est développé et sa convergence est démontrée. Cet algorithme demande relativement peu d'espace mémoire et son efficacité de calcul est très bonne.

Mots-clés: Affectation d'équilibre, transport en commun, inégalités variationnelles, décomposition simpliciale.

1 Introduction

In this paper we consider a transit equilibrium assignment problem which is defined as assigning the passenger flow on a congested transit network (or public transportation network) with asymmetric cost functions given a fixed origin-destination matrix. The passenger's travel behaviour is assumed to satisfy the user's equilibrium conditions (Wardrop, 1952). The transit network consists of zones where passengers are generated and attracted, the underlying road highway network and the transit lines that use it.

The transit assignment problem has attracted the interest of many researchers and practitioners, since in many cities the transit services are essential to ensuring the mobility of the inhabitants, either due to the increasing road congestion or the unavailability of the private car mode, as in developing countries. In order to maintain a suitable service, we have to understand how passengers travel in the transit network.

The problem, i.e., the transit assignment (equilibrium) problem, has been studied for more than one decade with many models proposed. Two issues should be considered when we study these models. The first issue is the assumption about the passenger travel behaviour, while the second issue is related to whether the transit network is congested or not, i.e., if the network is congested, then some equilibrium conditions should be assumed. Different combinations of these two issues lead to various models. For the case of uncongested transit networks, a good survey of the models is given by Spiess and Florian (1989). In this case, several assumptions have been proposed. The all-or-nothing assignment is a traditional assignment model which assumes that the passengers select shortest paths that are independent of passengers flows. The multipath assignment assumes that several paths are found first and the passengers are assigned proportionally to the costs of these paths. These methods do not consider the passenger behaviour at a bus stop carefully. In fact, to model the behaviour of the passengers waiting at a bus stop that is shared by several transit

lines, all the models assume certain behaviour that determine the passenger's choice. These choices are called strategies. Spiess (1984) and Spiess and Florian (1989) formulated a transit assignment model for an uncongested (i.e., fixed travel time) transit network based on the concept of the optimal strategy, as a linear programming and solved by a polynomial time algorithm. Therefore in their model, passengers always travel according to the optimal strategy, which results in a sort of multipath assignment. In a similar way, De Cea and Fernandez (1989) used a minimal route concept to formulate a linear program.

Nguyen and Pallottino (1985) characterized and studied in detail the notion of the strategy, by using a graph theoretical approach, and developed a concept of a hyperpath. A strategy corresponds to a hyperpath, i.e., a subnetwork containing all arcs of the corresponding strategy. As in Spiess's model, at a bus stop, passengers always board the first vehicle to arrive from the set of attractive lines of the hyperpath at that stop. The demand at a node is then assigned to the hyperpath in proportion to the fixed frequencies of lines in the attractive set of the hyperpath.

Although this class of models can be useful and efficient to study transit systems where the impact of congestion effects on the route choice is low, they are unsuitable in the simulation of transit flows in some cities, such as Shanghai where high congestion levels may be so severe that they affect the passenger's route choices. This is the second issue to consider.

Gendreau (1984) and De Cea and Fernandez(1988) extended the *TEAP* to include the consideration of congestion effects not only on the in-vehicle cost but also on the waiting time. The first formulates the waiting times with a queuing theoretical approach on a expanded network, while the second models the waiting time to be a direct function of the flow-dependent effective frequency; however, only a solution algorithm for a simplified version of the model is presented. Spiess and Florian (1989) gave a general version of a nonlinear model in which in-vehicle travel times (or generalized travel costs) are increasing functions of passenger flow (called "discomfort functions").

The main limitations of these approachs are that waiting times at stops are not affected by congestion and there are no cross effects of flows on the cost functions. A similar formulation was proposed by Nguyen and Pallottino (1985). They showed that, by explicitly defining the flow and structures of the hyperpath, the standard equilibrium traffic assignment model could be adapted to transit networks by substituting the hyperpath flow space for the path flow space. As in Spiess and Florian's model, the waiting times are assumed constant. It is noted that this adaptation is carried in the hyperpath flow space (not in the total arc flow space). In this sense, the transit equilibrium assignment model is different from the traditional traffic equilibrium model.

Recently, Wu, Florian and Marcotte (1991) formulated a transit equilibrium assignment model (WFM) as a variational inequality problem and solved it by using a linearized approximation method in the hyperpath flow space, while others considered various methods to deal with the congested travel behaviour (see Spiess and Florian, 1989, Nguyen and Pallottino, 1985, Gendreau, 1984, and De Cea and Fernandez, 1988). We note that Spiess and Florian (1989) developed a Frank-Wolfe method for a symmetric transit equilibrium assignment problem with a folw dependent arc time.

Since the WFM model is defined in the hyperpath space, the storage requirements for the hyperpath information are large, even though a restriction strategy is used to store only necessary hyperpaths. This storage increases as the number of zones and the size of the transit network increase.

The paper is devoted to the development of a method for transit equilibrium assignment problem by first converting the original variational inequality problem to an equivalent variational inequality problem in a new variable space and then using simplicial decomposition methods (Lawphongpanich and Hearn, 1984, and Pang and Yu, 1984) to solve it. The advantages of doing this are the following.

1. One need only store the limited number of extreme points in the space of total arc flows plus

one auxiliary variable (total waiting time due to the frequency delay). It requires much less memory than the WFM's algorithms which operate in the space of hyperpath flow;

2. On the other hand, it is very similar to the WFM method in that the quadratic subproblem is solved by a one-dimensional search and an extreme point is generated by a solution of a shortest hyperpath problem. Therefore, it is well suited for large scale transit equilibrium assignment problem , since it requires for less storage.

However, one potential difficulty is that the cost function in the new variable space may not be strongly monotone, while the assumption is needed for the convergence of the simplicial decomposition methods.

In the following, we will show that only the strong monotonicity of the cost functions in the space of the total arc flow is required, not in the whole variable space, for the convergence of the simplicial decomposition algorithm.

The rest of the paper is organized as follows. Section 2 presents the basic definitions and the formulations of the transit equilibrium assignment problem . In section 3, we give the basic simplicial decomposition algorithm for the equivalent transit equilibrium assignment problem and a discussion of the steps of the algorithm. Section 4 is devoted to the study of the convergence of the algorithm under usual assumptions. Finally, some conclusions are presented in Section 5.

2 Basic definitions and formulations

As discussed in the WFM model, the transit equilibrium assignment problem can be formulated as the following variational inequality problem : find $h^* \in \Omega_h$ such that

$$VIP(\Omega, S): \qquad S(h^*)^T(h - h^*) \geq 0, \quad h \in \Omega_h \tag{1}$$

where $S(h) \in R^m$ is a vector of hyperpath costs; $h \in R^m$ is a vector of hyperpath flows; $h^* \in R^m$ is a vector of equilibrium hyperpath flows and Ω_h is a feasible region satisfing the following

$$\sum_{k \in K_i} h_k = g_i, \quad i \in I,$$

$$h_k \geq 0, \quad k \in K_i, i \in I,$$

where h_k is the flow along hyperpath k, g_i is the travel demand from origin p to destination q, $(p,q) = i, K_i$ is the set of hyperpaths from origin p to destination q, and I is the set of origin-destination pairs.

It is known (Nguyen et al., 1985, Wu, Florian and Marcotte, 1991) that

$$S(h^*)^T(h - h^*) = s(v^*)^T(v - v^*) + W^T(h - h^*) \tag{2}$$

where $s(v) = (s_a(v))$ is a vector of arc cost, $v = \delta h$ (δ is the hyperpath-arc incidence matrix) and $W = (W_k)$ is a vector of hyperpath waiting time due to the frequency delay, i.e.,

$$v_a = \sum_{i \in I} \sum_{k \in K_i} \delta_{ak} h_k$$

and

$$W_k = \sum_{a \in A} \delta_{ak} \frac{1}{\sum_{a' \in FS_k(t(a))} f_{a'}}$$

where v_a is the total flow on arc a, W_k is the waiting time on the hyperpath k due to the frequency delay in the transit network, A is the set of arcs in the transit network, f_a is the bus frequency associated with arc a, $t(a)$ is the tail of arc a and $FS_k(t)$ is the set of forward arcs at the node t.

According to (2), it appears that one has to use hyperpath related information. In fact, this is not always necessary. Consider the following new variable space $\hat{\Omega}$ defined by

$$\hat{\Omega} = \{(v,u) : (v,u) = (\delta, W)^T h, \ \forall h \in \Omega_h\}. \tag{3}$$

Then we have the following proposition.

Proposition 2.1 *Assume that the set* $\Omega_v = \{v : v = \delta h, \forall h \in \Omega_h\}$ *is a bounded polyhedron. The new variable space* $\hat{\Omega}$ *is also a bounded polyhedron.*

Proof. First, we show that $\hat{\Omega}$ is a polyhedron. According to (3), we know that $\hat{\Omega}$ is the region obtained from a polyhedron via a linear transformation (δ, W). Therefore $\hat{\Omega}$ is a polyhedron. Since we assume that $\Omega_v = \{v : v = \delta h, \forall \ h \in \Omega_h\}$ is bounded, W is a nonnegative constant vector and Ω_h is bounded, then $\hat{\Omega}$ is also bounded. □

By the definition, $\hat{\Omega}$ is also a convex set and it can be expressed as a linear combination of its extreme points, i.e.,

$$Co(e_1, e_2, ...) = \hat{\Omega}$$

where Co denotes the convex hull of the extreme points $(e_1, e_2, ...)$ of $\hat{\Omega}$.

According to (3), we can see that every distinct extreme point in $\hat{\Omega}$ corresponds to a distinct extreme point in Ω_h. The reverse is not necessarily true since (δ, W) is, in general, singular. Thus a set of extremes in Ω_h may correspond to a single point in $\hat{\Omega}$.

The proceeding Proposition indicates that one may reformulate the problem in the space of total arc flow v and the auxiliary variable u, by using successive hyperpath computation to generate extreme points. The space $\hat{\Omega}$ is gradually constructed by the convex hull of the generated points of $\hat{\Omega}$. The following proposition exploits this property.

Proposition 2.2 *(1) is equivalent to find* $(v^*, u^*) \in \hat{\Omega}$ *such that*

$$s(v^*)^T(v - v^*) + (u - u^*) \geq 0, \quad \forall \ (u, v) \in \hat{\Omega} \tag{4}$$

where $\hat{\Omega}$ *is the new variable space.*

Proof.

$$s(v^*)^T(v - v^*) + (u - u^*) \geq 0$$

implies that

$$s(\delta h^*)^T(\delta h - \delta h^*) + W^T(h - h^*) \geq 0$$

which again implies that

$$S(h^*)^T(h - h^*) + W^T(h - h^*) \geq 0. \qquad \square$$

The following proposition is a direct consequence of a well-known theorem in variational inequality problem., i.e.,

Proposition 2.3 *(4) is equivalent to the following problem:*

$$\min_{(v,u)\in\hat{\Omega}} G(v,u) = -s(v)^T(\hat{v} - v) - (\hat{u} - u) \tag{5}$$

where $(\hat{v}, \hat{u})$ *is a solution of the following lincar program:* $\min_{(x,y)\in\hat{\Omega}} s(v)^T(x - v) + (y - u)$ *and* $G(v^*, u^*) = 0$ *if and only if* (v^*, u^*) *is the solution of the variational inequality problem (4).*

3 The basic algorithm

In the following, we give a simplicial decomposition algorithm (SDA) and discuss in detail the computations required at each step.

SDA

- **Step 1** (Initialization) Find an extreme point (e_v^1, u_u^1) of $\hat{\Omega}$ and define

$$E^1 = \begin{pmatrix} E_v^1 \\ E_u^1 \end{pmatrix} = (e^1) = \begin{pmatrix} e_v^1 \\ e_u^1 \end{pmatrix}.$$

Let $\bar{G}^1 = \infty$, β be a positive parameter and $l = 1$. Let $\hat{\Omega}^l$ be the convex hull of the columns of E^l and ϵ^l be a error such that $\epsilon^l \to 0$ as $l \to \infty$.

- **Step 2** (Solve the restricted variational inequality problem) Find $(v^{l+1}, u^{l+1}) \in \hat{\Omega}^l$ such that

$$s^{l+1}(v^{l+1})^T(v - v^{l+1}) + (u - u^{l+1}) \geq -\epsilon^l \tag{6}$$

where

$$s^{l+1}(v) = s(v^l) + \frac{B(v^l)}{\alpha}(v - v^l).$$

- **Step 3** (Solve the shortest hyperpath)

$$\min_{(e_v, e_u) \in \hat{\Omega}} s(v^l)^T e_v + e_u. \tag{7}$$

- **Step 4** (Stopping criterion) **if** $G(v^l, u^l)$=0 , **then** stop. (v^l, u^l) is the solution.

- **Step 5** (Generate E^{l+1}) **if** $G(v^l, u^l) \geq \bar{G}^l - \beta$, **then** $E^{l+1} = E^l \bigcup \{ \begin{smallmatrix} e_v \\ e_u \end{smallmatrix} \}$;

if $G(v^l, u^l) < \bar{G}^l - \beta$, **then** $E^{l+1} = E^l \backslash D^l \bigcup \{ \begin{smallmatrix} e_v \\ e_u \end{smallmatrix} \}$;

where D^l= columns of E^l with zero weight in the expression of (v^l, u^l) as a convex combination of column of E^l. Let

$$\bar{G}^{l+1} = \min\{\bar{G}^l, G^l(v^l, u^l)\}.$$

$l = l + 1$. Goto **Step 2**.

In **Step 1**, the (e_v^1, e_u^1) can be found by the solution of a shortest hyperpath problem where the arc cost is set to be $s(0)$. In **Step 2**, a solution of (6) may lead to a sequence of quadratic minimization

problems in $\hat{\Omega}^l$. A possible algorithm called the linearized method may be the following. For an initial solution $(v^1, u^1) \in \hat{\Omega}^l$, solve $(v^{k+1}, u^{k+1}) = H(v^k, u^k)$ for $k = 1, 2, ...$ where $H(v^k, u^k)$ is the solution of the following quadratic programming:

$$\min_{(v,u)\in\hat{\Omega}^l} s(v^k)^T(v - v^k) + \frac{1}{2\alpha}(v - v^k)^T B(v^k)(v - v^k) + (u - u^k)$$

which is equivalent to the following problem

$$\min_{1^T\lambda=1,\lambda\geq 0} s(v^k)^T(E_v^l\lambda - v^k) + \frac{1}{2\alpha}(E_v^l\lambda - v^k)^T B(v^k)(E_v^l\lambda - v^k) + (E_u^l\lambda - u^k) \tag{8}$$

where α is a positive parameter. Thus the above problem can be solved by an one-dimensional search algorithm. In the **Step 3**, as in Step 1, the shortest hyperpath problem can be solved in a polynomial time (see Spiess and Florian, 1989 and Nguyen and Pallottino, 1985).

4 Convergence analysis

In this section, we discuss the convergence of the the simplicial decomposition algorithm (SDA) for the equivalent variational inequality problem . We define steps which compute the shortest hyperpath problems as major cycles (also called the subproblems) and steps of solutions of the restricted variational inequality problem as minor cycles (also called the master problem).

Proposition 4.1 *If each minor cycle is well defined, then the SDA converges to a unique solution* (v^*, u^*).

Proof. For the convergence see Lawphongpanich and Hearn (1984). For the uniqueness see Wu, Florian and Marcotte (1991). □

Now we consider the global convergence of the master program, i.e., the restricted variational inequality problem and show that the master program is well defined.

Theorem 4.1 (Convergence Theorem of The Master Problem) *Let $\hat{\Omega}^l$ be a nonempty compact and convex set. Let $s(v)$ be continuous mapping from R^m into itself. Let $B(v)$ be a symmetric positive definite matrix from R^m to $R^m \times R^m$. Then there exists a symmetric positive definite matrix $\overline{B}$ with $B(v) - \overline{B}$ positive semi-definite for all $v \in \hat{\Omega}^l$. If there is a parameter $0 < \theta < 1$ such that, for all $(v', u'), (v'', u'') \in \hat{\Omega}^l$,*

$$||\overline{B}^{-1}\{\alpha[s(v') - s(v'')] - B(v')(v' - v'')\}||_{\overline{B}} \leq \theta||v' - v''||_{\overline{B}} \tag{9}$$

then the sequence of iterates $\{v^k, u^k\}$ generated by the linearized method converges to a unique solution of the restricted variational inequality problem for any initial vector $(v^1, u^1) \in \hat{\Omega}^l$.

Proof. The existence of $\overline{B}$ is obvious since $\overline{B}$ can be selected such that

$$\overline{B} = \lambda_{min} I$$

where λ_{min} is the minimum eigenvalue of $B(v)$ and I is a corresponding identity matrix.

Consider following two inequalities

$$[s(v^{k-1}) + \frac{1}{\alpha}B(v^{k-1})(v^k - v^{k-1})]^T(v^{k+1} - v^k) + (u^{k+1} - u^k) \geq 0$$

and

$$[s(v^k) + \frac{1}{\alpha}B(v^k)(v^{k+1} - v^k)]^T(v^k - v^{k+1}) + (u^k - u^{k+1}) \geq 0.$$

Adding them up and rearranging the term, we have

$$(v^{k+1} - v^k)^T \frac{1}{\alpha}B(v^k)(v^{k+1} - v^k)$$

$$\leq [s(v^{k-1}) + \frac{1}{\alpha}B(v^{k-1})(v^k - v^{k-1}) - s(v^k)]^T(v^{k+1} - v^k)$$

$$\leq ||\overline{B}^{-1}\{(s(v^{k-1}) - s(v^k) - \frac{1}{\alpha}B(v^{k-1})(v^k - v^{k-1})\}||_{\overline{B}}||v^{k+1} - v^k||_{\overline{B}}.$$

Thus

$$||v^{k+1} - v^k||_{\overline{B}} \leq \theta ||v^k - v^{k-1}||_{\overline{B}}.$$

By the standard contraction theory, the algorithm is contracting and hence we have

$$\lim_{k \to \infty} v^{k+1} = v^k = v^*,$$

and let

$$\lim_{k \to \infty} u^k = u^*.$$

From

$$[s(v^{k-1}) + \frac{1}{\alpha} B(v^{k-1})(v^k - v^{k-1})]^T (v - v^k) + (u - u^k) \geq 0,$$

we obtain

$$s(v^*)^T (v - v^*) + (u - u^*) \geq 0.$$

By definition, (v^*, u^*) is a solution of the restricted variational inequality problem . According to Wu, Florian and Marcotte (1991), it is unique in $\hat{\Omega}$. □

We note that

- Various choices of $B(v)$ would lead to different methods. For example, if $B(v)$ is a constant symmetric and positive definite matrix, the the method is the projection method; if $B(v)$ is a diagonal matrix of $\nabla s(v)$, then the method is the linearized Jacobi method.
- The sufficient conditions for (9) have been provided in Wu, Florian and Marcotte (1991) where δ is set to a identity matrix for this our problem if the linerized Jacobi method is used. The conditions are the strong monotonicity, Lipschitz continuity of $s(v)$ and a parameter requirement.

5 A method for Step 2

In the following, we present an algorithm to solve the restricted variational inequality problem (6). The problem (6) can be formulated as follows (superscript l for E is dropped for simplicity): find $\lambda^* \in \Omega_\lambda = \{\lambda : 1^T\lambda = 1, \lambda \geq 0\}$ such that

$$s(E_v\lambda^*)^T(E_v\lambda - E_v\lambda^*) + (E_u\lambda - E_u\lambda^*) \geq -\epsilon \quad \forall\lambda \in \Omega_\lambda, \tag{10}$$

i.e.,

$$[s(E_v\lambda^*)^T E_v + E_u](\lambda - \lambda^*) \geq -\epsilon, \quad \forall\lambda \in \Omega_\lambda. \tag{11}$$

The solution of the problem leads to a sequence of simpler quadratic problems. For $\lambda^0 \in \Omega_\lambda$, solve $\lambda^{k+1} = H(\lambda^k)$ for $k = 0, 1, 2, ...$, until some criteria are satisfied where $H(\lambda^k)$ is the solution of the following quadratic program:

$$\min_{\lambda\in\Omega_\lambda} [s(E_v\lambda^k)^T E_v + E_u](\lambda - \lambda^k) + \frac{1}{2\alpha}(\lambda - \lambda^k)^T E_v^T B(v^k) E_v(\lambda - \lambda^k), \tag{12}$$

where $B(v^k)$ is assumed to be the diagonal of $\nabla s(v^k)$.

We define

$$\hat{s}(\lambda^k)^T = s(E_v\lambda^k)^T E_v + E_u = s(v^k)^T E_v + E_u \tag{13}$$

and

$$\hat{B}(\lambda^k) = E_v^T B(v^k) E_v. \tag{14}$$

Now we consider one stopping criterion. Let

$$\Psi^k = \min_{\lambda\in\Omega_\lambda} \hat{s}(\lambda^k)^T(\lambda - \lambda^k). \tag{15}$$

If $\Psi^k \geq -\epsilon$, then stop. We find $H(\lambda^k)$. The linear program (15) is extremely easy to solve, since the solution can be found directly. The problem (12) is rewritten by the following:

$$\min_{\lambda \in \Omega_\lambda} \hat{s}(\lambda^k)^T(\lambda - \lambda^k) + \frac{1}{2\alpha}(\lambda - \lambda^k)^T \hat{B}(\lambda^k)(\lambda - \lambda^k). \tag{16}$$

The solution of (16) leads to a one-dimensional search algorithm and a calculation of the inverse of $\hat{B}(\lambda^k)$. Since the matrix $\hat{B}(\lambda^k)$ is not very large in practice, the calculation of the inverse is not expensive. Now we take the K-T condition of (16) and obtain

$$\hat{s}(\lambda^k) + \frac{1}{\alpha}\hat{B}(\lambda^k)(\lambda - \lambda^k) - 1^T\mu - \beta = 0, \tag{17}$$

$$1^T\lambda = 1, \tag{18}$$

$$\lambda \geq 0, \tag{19}$$

$$\lambda^T\beta = 0, \tag{20}$$

$$\beta \geq 0. \tag{21}$$

From (17), we have

$$\lambda = \max\{\lambda_k + \alpha\hat{B}(\lambda^k)^{-1}(1^T\mu - \hat{s}(\lambda^k)), 0\}, \tag{22}$$

and

$$\beta = \max\{\hat{s}(\lambda^k) + \frac{1}{\alpha}\hat{B}(\lambda^k)\lambda_k - 1^T\mu, 0\}. \tag{23}$$

As a consequence we show the following results.

Proposition 5.1 *(22) and (23) satisfy (17).*

Proof. Case 1. We have

$$\lambda_k + \alpha\hat{B}(\lambda^k)^{-1}(1^T\mu - \hat{s}(\lambda^k)) > 0.$$

Then by (22), we obtain

$$\lambda = \lambda_k + \alpha \hat{B}(\lambda^k)^{-1}(1^T\mu - \hat{s}(\lambda^k))$$

which implies that

$$\hat{s}(\lambda^k) - \frac{1}{\alpha}\hat{B}(\lambda^k)(\lambda - \lambda^k) - 1^T\mu = 0.$$

From (23), $\beta = 0$. Thus (17) is satified.

Case 2.

$$\lambda^k + \alpha \hat{B}(\lambda^k)^{-1}(1^T\mu - \hat{s}(\lambda^k)) \leq 0.$$

By (22), $\lambda = 0$. According to (23),

$$\beta = \hat{s}(\lambda^k) - \frac{1}{\alpha}\hat{B}(\lambda^k)\lambda_k - 1^T\mu$$

which implies that (17) is satisfied. □

Proposition 5.2 *(22) and (23) satisfy (20).*

Proof. This is straightforward to verify. □

Thus only constraint that must be satisfied is constraint (18). We define

$$\Gamma(\mu) = 1^T\lambda(\mu). \tag{24}$$

Then Γ is nondecreasing, monotone and piecewise linear. Therefore one only needs to carry out a one-dimensional type search algorithm to find μ^* such that

$$\Gamma(\mu^*) = 1. \tag{25}$$

Such a one-dimensional algorithm can be found in Wu (1991). As long as μ^* is determined, we use (22) to find λ^*. We note that the method to solve the quadratic nonseparable program (10) is more general than one in Wu (1991) where a quadratic program is separable.

6 Conclusions

In this paper, we develop a simplicial decomposition algorithm (SDA) for the transit equilibrium assignment problem . The problem has a different structure, i.e., it is only defined in the hyperpath space. Nevertherless, the simplicial decomposition approach is adopted for this problem in the space of total arc flow and the auxiliary variable. It represents a computationally tractable way of solving large scale transit equilibrium assignment problem. A proof of the convergence of the SDA is also given, which is adapted from a similar idea of Lawphongpanich and Hearn (1984) and Pang and Yu (1984). We intend to report in near future computational results with large scale problems.

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